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The Poverty-Adjusted Life Expectancy index: a consistent aggregation of the quantity and the quality of life *

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Abstract

Poverty and mortality are arguably the two major sources of well-being losses. Most mainstream measures of human development capturing these two dimensions aggregate them in an ad-hoc and controversial way. In this paper, we propose a new indicator aggregating the poverty and the mortality observed in a given period, which we call the poverty-adjusted life-expectancy (PALE). This indicator is based on a single normative parameter that transparently captures the trade-off between well-being losses from being poor or from being dead. We first show that PALE follows naturally from the expected life-cycle utility approach a la Harsanyi (1953). Empirically, we then proceed to between countries or across time comparisons and focus on those situations in which poverty and mortality provide conflicting evaluations. Once we assume that being poor is (at least weakly) preferable to being dead, we show that about a third of these conflicting comparisons can be unambiguously ranked by PALE. Finally, we show that our index naturally defines a new and simple index of multidimensional poverty, the expected deprivation index, which aggregates poverty and premature mortality in a consistent way.

JEL: D63, I32, O15.

Keyworks: Well-being index, Human development index, Multidimensional poverty, Poverty, Mortality.

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1 Introduction

There is a long-standing tradition looking for an indicator able to track the level of human development in a society (Hicks and Streeten, 1979; Stiglitz et al., 2009; Fleurbaey, 2009). By measuring well-being in a given period, this measure would allow comparisons of human development across countries and across time. For this purpose, simple monetary measures, such as GDP per head, have been heavily criticized, essentially on two accounts.[1] First, income aggregates such as GDP are typically blind to the distribution of consumption across the population. This distributional concern has lead to the design and adoption of income poverty measures (see, e.g., World Bank (2015)). Second, key aspects of human well-being, such as health or education, are virtually impossible to translate into monetary values. As a result, monetary measures do not provide a sufficient informational basis to account for the multi-dimensional nature of human development. This multi-dimensional concern has lead to the adoption of dashboards of indicators, such as the 17 Sustainable Development Goals (SDG) adopted in 2015 by the UN.

Given these limitations, many authors proposed several ways to aggregate different dimensions of well-being into a single indicator of human development that could be readily applied, for instance to assess a country's performance at promoting well-being or to evaluate policies that imply trade-offs between different dimensions, e.g. safety regulations, environmental or health related policies. Among these indicators, one finds the Human Development Index (HDI) (UNDP, 1990), the Level of Living Index (Drewnowski and Scott, 1966) or the Physical Quality of Life Index (Morris, 1978). Echoing the distributional concern, some of them focus on deprivations, like the Global Multidimensional Poverty Index (Alkire et al., 2015) or the Human Poverty Index (Watkins, 2006). These summary measures provide a rough yardstick of human development, which is arguably easier to communicate than a full list of various indicators. Crucially, they have the potential to solve the partial ranking of societies yielded by a dashboard of indicators, when one society performs better along one dimension than another society but not along another. A dashboard cannot compare two societies when two or more dimensions are "in conflict".

All these composite indices are subject to the same fundamental criticisms (Ravallion, 2011a,b; Ghislandi et al., 2019). First, the selection of the appropriate indicator in each dimension and the choice of the aggregation function are often arbitrary and they do not follow from a defensible notion of individual well-being. Second, the system of weights embedded in the aggregation function is itself often arbitrarily selected, for instance by giving an equal weight to each dimension. Such weights are therefore not related to the trade-offs that individuals would make between these dimensions, and cannot be taken as representative of human well-being.[2] Together, these critics are devastating. Indeed, the full ranking of societies yielded by composite indices is of little value if the trade-off they make between "conflicting" dimensions is not meaningful or arbitrary. Finally, the value of a summary indicator also depends on how quickly it can be grasped. Unfortunately, these indicators, originally conceived as pragmatic ordinal indicators, do not typically offer a simple interpretation,

[1] Another important critic relates to sustainability of the well-being achieved in a particular period.

[2] More fundamentally, different individuals may make different trade-offs, which implies that no system of weights can be completely consensual.

that can be easily communicated.

These critics are so compelling that a number of scholars argue in favor of reducing the informational basis to a unique dimension, such as health (Hicks and Streeten, 1979), thereby avoiding the need to choose a particular aggregation process. Among health indicators, a prominent and easily interpretable indicator is the life-expectancy at birth, which can also be adapted in order to account for the distributional concern (Silber, 1983; Ghislandi et al., 2019; Gisbert, 2020). According to these authors, the cost of reducing the number of dimensions accounted for might not be that high, as not all dimensions carry the same importance for human well-being. Moreover, some dimensions can be considered mostly as "inputs" for well-being, rather than "outcomes". For instance, sanitation may be considered as an input in some health production function.

In this paper, we propose to measure human well-being using the poverty-adjusted life-expectancy (PALE), a new summary index that aggregates well-being losses resulting from the poverty and mortality observed in a given period. This summary index makes substantial progress on the criticisms identified above. First, the aggregation of poverty and mortality is normatively grounded on the expected life-cycle utility, the measure of social welfare proposed by Harsanyi (1953). Second, even though our index relies on some weight, we show that the index improves on the partial ranking yielded by the separate dimensions, as long as one considers that being poor is not worse than being dead. Under this assumption, some pairs of societies are unambiguously compared by our index, even when the two dimensions are "in conflict", for instance if one society has less poverty but higher mortality than the other. A necessary and sufficient condition for unambiguous comparisons is that the index makes the same comparison for the two extreme values for its weight. Third, the interpretation of our index under these two extreme values is straightforward. Under one extreme value, our indicator is the life-expectancy at birth. Under the other extreme value, our indicator is the poverty-free life-expectancy at birth (Riumallo-Herl et al., 2018), i.e. it is the number of years of life that a newborn expects to live out of poverty, if she assumes poverty and mortality remain constant over time.

There are good reasons to focus on poverty and mortality when measuring human development. First, poverty and mortality are arguably the two major sources of welfare losses: poverty entails welfare losses by reducing the quality of life while mortality entails welfare losses by reducing the quantity of life. Prominent scholars in welfare economics such as Deaton and Sen have dedicated a large part of their work to the study of poverty and mortality (Deaton, 2013; Sen, 1998). Unsurprisingly, the first two Sustainable Development Goals of the UN are directly related to poverty while the third one refers to mortality.[3] Second, focussing on poverty and mortality naturally reflects distributional concerns as they are the worst possible outcomes associated with consumption and health. Finally, note that the poverty status we consider here could also be a measure resulting from some aggregation of different dimensions of the quality of life.

Our index is based on a simplified version of social welfare a la Harsanyi (1953). According to Harsanyi, social welfare in a given period can be understood as the life-cycle utility expected by a newborn when drawing at random a life that re-

[3] The first two SDGs are entitled "No Poverty" and "Zero Hunger", while the majority of the indicators in the third "Good Health and Well-being" section refer to some form of mortality.

flects the outcomes observed in that particular period. Our main simplification is to consider a binary quality of life: in any period, an individual is either poor or non-poor.[4] In this context, life-cycle utility is the sum of period utilities over the lifespan, where period utility takes two values, one high when non-poor and one low when poor. Our index therefore normalizes the expected life-cycle utility when one expects, throughout her lifetime, to be confronted to the poverty and mortality prevailing in the current period. We call this index "poverty-adjusted life-expectancy", because in such stationary society, this index simply counts the number of periods that a newborn expects to live but weighs down the periods that she expects to live in poverty. Mathematically, our index is obtained by multiplying life-expectancy at birth by a factor one minus the fraction of poor, where the fraction of poor is weighed down. This (normative) weight, the value of which lies between zero and one, corresponds to the fraction of the period utility lost when poor. When being poor has no utility cost, this weight takes the value zero. When being poor is as bad as losing one year of life, this weight takes the value one. As we make clear later, our index is not a projection or a forecast of the average life-cycle utility of the cohort born in a particular period, implying that it cannot in general be interpreted as the expected life-cycle utility of a newborn, unless the society is stationary. However, even when mortality is selective and affects predominantly poor people, we also show that our index still constitutes a meaningful way to aggregate the two sources of welfare losses observed in a particular period.

Our index improves on the partial ranking provided by a dashboard considering poverty and mortality separately. More precisely, there are pairs of societies that cannot be compared using a dashboard, while our index provides the same strict comparison for all plausible values of its weight. For instance, consider two societies A and B where B has a higher fraction of poor but a higher life-expectancy at birth, i.e. lower mortality. Suppose that the situation is such that one may expect to spend more periods in poverty in B than in A *but* also more periods out of poverty in B than in A, as people live longer in society B. It is easy to show that life-cycle utility is larger in B, regardless of the weight given to periods of poverty, because individuals on average live more periods of both types in B. Hence, provided that being poor is not worse than being dead, our index unambiguously ranks A and B, which a dashboard approach is unable to do. As a result, our index increases the set of pairs of societies that can be unambiguously compared.

Empirically, we combine datasets provided by the World Bank data on income poverty (PovCalNet) and internationally comparable dataset on mortality data (the Global Burden of Disease) from 1990 to 2015. Again assuming that one year spent in poverty is (weakly) preferred to one year of life lost, we show that PALE is able to solve a non-trivial number of ambiguous comparisons across time or between countries. For instance, when comparing all possible pair of countries in each year, across all years, there are about 21 percent of such comparisons for which mortality and poverty move in opposite directions. Out of these ambiguous cases, PALE is able to solve 33 percent of them. We also investigate the evolution of each country in the dataset, by comparing the situation in a particular year to that prevailing five

[4] Clearly, we do not claim that our index is superior to Harsanyi's approach, but it is a plausible measure of expected life-cycle utility when poverty is considered to be the main factor reducing the quality of life.

years earlier. We find that, out of 28 percent of ambiguous comparisons, PALE is able to solve 38 percent of them.

Finally, we propose an adapted version of our index that explicitly addresses distributional concerns about unequal lifespans. We define a new indicator of multidimensional poverty that captures deprivations in the quality and quantity of life. If being deprived in the quality of life can be equated to being poor, being deprived in the quantity of life requires the introduction of a normative age threshold below which one is considered as deprived, i.e. a definition of premature mortality. This new index, which we call the expected deprivation index (ED), is a weighted sum of the number of years that a newborn expect to loose prematurely and the number of years she expects to spend in poverty, using the same weight as in PALE. (Again, these expectations assume that the newborn is confronted throughout her lifespan to the poverty and mortality observed in the current period.) We show that this index also increases the set of pairs that can be unambiguously compared when considering the dimensions separately. In its spirit, ED is similar to the Generated Deprivation index recently proposed by Baland et al. (2021), and they are in fact equal in stationary societies. We show that ED is more reactive to contemporaneous policies (e.g. in the case of permanent mortality shocks). Moreover, ED is less data demanding and subject to a simpler interpretation than Generated Deprivation.

This research thus contributes to the literature on human development measures by proposing a multidimensional indicator that focusses on the worst possible outcomes. The novelty is to define a multidimensional index of well-being in a given period that (i) captures the two main sources of welfare losses, (ii) aggregates different dimensions in a theoretically sound way, (iii) provides a robust ranking for a set of comparisons for which its two dimensions conflict, (iv) has a direct and intuitive interpretation and (v) can readily be applied to the available data. Moreover, we show that the PALE index is related to a new index of multidimensional poverty, Expected Deprivation, that enjoys the same advantages and can complement PALE if one is concerned with unequal lifespans.

The poverty-adjusted life-expectancy is reminiscent of several indicators proposed in health economics, like the quality-adjusted life-expectancy (QALE) or the quality-adjusted life year (QALY).[5] Both account for the quality and quantity of life, by weighting down the quantity of life for periods with low quality. They have been developed following the method of Sullivan (1971) and we show that these approaches directly follow from the expected life-cycle utility approach in stationary societies. Our index however accounts for another important dimension of well-being than health, which is poverty. Also, PALE takes advantage of the existence of the well-established concept of a poverty threshold, which splits the population into poor and non-poor, thereby transforming the quality of life into a binary variable. This transformation is key to the simple interpretation of our index. There is, to the best of our knowledge, no immediate equivalent of such threshold in health economics.

There exist other indicators of a society's well-being which are, in theory, arguably much superior to ours. Yet, these indicators either rely on techniques that are not mature yet, require many arbitrary assumptions or cannot be readily applied for

[5]See for instance Whitehead and Ali (2010) for an economic interpretation of QALYs, or Heijink et al. (2011); Jia et al. (2011) for applications of the QALE index to comparisons of health outcomes across populations.

all countries using existing data. For instance, Becker et al. (2005) and Jones and Klenow (2016) follow more sophisticated versions of Harsanyi's expected life-cycle utility approach. Alternatively, Fleurbaey and Tadenuma (2014), in the case of well-being, or Decancq et al. (2019), for poverty, propose to aggregate different dimensions using individual preferences.[6] Also, there is a large litterature investigating the weights to be given to different dimensions of well-being (Benjamin et al., 2014; Decancq and Lugo, 2013). However, this literature has not reached full maturity, or cannot be applied on a large scale due to data constraints.

The remainder of the paper is organized as follows. In Section 2, we present the theory supporting our PALE index and provide some empirical implications. In Section 3, we present the theory supporting our ED index; We provide some concluding comments in Section 4.

2 A transparent index of welfare

Our objective is to propose a simple indicator allowing to measure and compare the level of human development of different societies in a given period. In particular, we would like this indicator to aggregate two major sources of welfare losses: mortality – which reduces the quantity of life – and poverty – which reduces the quality of life – occuring during that period. This aggregation should follow from the way individuals aggregate these losses and therefore be related to life-cycle preferences.

The rationality requirements of decision theory provide a structure on admissible life-cycle preferences. Rational preferences over streams of consumption have been axiomatized by Koopmans (1960) and later generalized by Bleichrodt et al. (2008). Such preferences must be represented by a discounted utility function, which aggregates these streams as a discounted sum of period utilities

$$U = \sum_{a=0}^{d} \beta^a u(c_a) \tag{1}$$

where $d \in \mathbb{N}$ is the age at death, $\beta \in [0, 1]$ is the discount factor, c_a is consumption at age a and u is the period utility function.

Building on this representation of preferences, Harsanyi (1953) proposes to measure the welfare of a society by aggregating life-cycle utilities over the whole society. According to Harsanyi (1953), behind the veil of ignorance, each newborn faces a lottery whereby she ignores whether she will be poor and for how long, or whether she will be the victim of a premature death. When evaluating her life-cycle utility, she considers drawing at random the life of any individual in that society. Following the formulation of Jones and Klenow (2016), her expected life-cycle utility is given by

$$EU = \mathbb{E} \sum_{a=0}^{a^*-1} \beta^a u(c_a) S(a), \tag{2}$$

where $S(a)$ is the (unconditional) probability that the newborn survives to age a, a^* is the maximal lifespan one can reach and the expectation operator $\mathbb{E}$ applies to the uncertainty with respect to consumption c_a. The period utility when being dead

[6] The limits of these different approaches are reviewed in Fleurbaey (2009).

is normalized to zero. As a result, mortality is valued through its opportunity cost: death reduces the number of periods during which a newborn expects to consume.

Although this approach has solid theoretical foundations, it does not seem that the indicator defined by Eq. (2) could be directly used as a summary measure of human development. Indeed, this indicator first requires the choice of a particular mathematical expression for the period utility function u. Moreover, the trade-off that this indicator makes between the quantity and quality of life, which depends on the definition of u, remains relatively obscure. And, finally, this indicator, being expressed in utility-units, does not lend itself to a direct interpretation.

2.1 The PALE index

In order to improve on these issues, we consider two assumptions that simplify Eq. (2) into a simple index of human development.

Our first simplifying assumption is to ignore discounting, i.e. take $\beta = 1$. We argue that such assumption is necessary in order to assign equal weights to all individuals, regardless of their age. Indeed, Eq. (2) equates a society's welfare in a given period to the expected life-cycle utility of individuals born in that period. Clearly, the expected life-cycle utility of newborns is related to the society's welfare in a given period only when one assumes that their expected lives reflect at each age the outcomes observed for individuals of that age during the period considered. When discounting with a factor less than one, we give less weight to the outcomes of older individuals. As a result, to give the same weight to individuals of different ages implies $\beta = 1$.

Our second simplifying assumption is to transform consumption into a binary variable, i.e. c_a can be either being non-poor (NP) or being poor (P). This is obviously a strong assumption because we ignore the impact on period utility of consumption differences inside these two categories. However, we argue that this assumption reflects the distributional concern, i.e. the desire to evaluate a society's development by focussing on the fate of its least well-off individuals. We believe this assumption is the price to pay when one wishes to focus on poverty as the main source of welfare losses, rather than other more general determinants of the quality of life.[7]

Taken jointly, these two assumptions require the use of a simple indicator, which we call the poverty-adjusted life-expectancy (PALE). Our second assumption implies $\mathbb{E}u(c_a) = \pi(a)u_P + (1 - \pi(a))u_{NP}$ where $u_{NP} = u(NP)$, $u_P = u(P)$ and $\pi(a)$ is the probability to be poor at age a conditional on being alive at age a. As, by definition, life-expectancy at birth is $LE = \sum_{a=0}^{a^*-1} S(a)$, we can rewrite Eq. (2) as

$$EU = u_{NP}LE - (u_{NP} - u_P) \sum_{a=0}^{a^*-1} S(a)\pi(a). \tag{3}$$

In Section 2.3, we derive a result showing that these two assumptions are sufficient to define PALE when aggregating the welfare losses coming from poverty and mortality in a given period. Here we provide a simple illustration showing how these

[7]This assumption is also used by Decerf et al. (2020) in a study of the poverty and mortality effects of the Covid-19 pandemic. These authors compare the relative sizes of poverty and mortality shocks, whereas we derive here an indicator of well-being.

two assumptions naturally lead to our index under a third assumption of "independence". Let us assume that the conditional probability of being poor at each age a is a constant equal to the fraction of poor in the population, i.e. $\pi(a) = H$ for all $a \in \{0, \ldots, a^* - 1\}$ where H is the head-count ratio. Clearly, this independence assumption does not hold when mortality is selective, for instance when the poor die younger than the non-poor. We discuss this limitation in more details in Section 2.3. Under this assumption, we can normalize Eq. (3) as

$$\frac{EU}{u_{NP}} = LE\Big(1 - \underbrace{\frac{u_{NP} - u_P}{u_{NP}}}_{\theta} H\Big).$$

This last expression defines the poverty-adjusted life-expectancy index:

$$PALE_\theta = LE(1 - \theta H). \tag{4}$$

The parameter $\theta \in [0, 1]$ captures the fraction of period utility lost when a non-poor individual becomes poor in a given period.[8] Importantly, this parameter directly captures the trade-off between poverty and mortality. Indeed, as the period utility of being dead u_D is normalized to zero, we have $\frac{1}{\theta} = \frac{u_{NP} - u_D}{u_{NP} - u_P}$. Hence, the ratio $\frac{1}{\theta}$ measures, for a non-poor individual, the number of periods in poverty that are equivalent to being dead for one period.

PALE has a simple expression: its first factor measures life-expectancy, whereas its second factor captures the fall in the quality of life due to poverty. This reduction depends on the value assigned to the parameter θ. When $\theta = 0$, that is becoming poor does not affect the quality of life, PALE reduces to life-expectancy at birth. When $\theta = 1$, that is becoming poor brings the quality of life to zero, PALE still depends on life-expectancy. In this particular case, PALE corresponds to the Poverty Free Life Expectancy (PFLE), an indicator proposed by Riumallo-Herl et al. (2018),[9] which measures the number of years that an individual expects to live free from poverty, when confronted throughout her life to the poverty and mortality observed in the period. For other values for θ, PALE corresponds to the number of years of life free from poverty that provides the same life-cycle utility as that expected by a newborn (when assuming again that poverty and mortality remain constant).

PALE aggregates a measure of mortality, LE, with a measure of poverty, H, in a way consistent with life-cycle preferences. This is a progress over most composite indices, but PALE also relies on a normative parameter that weights these two dimensions. Thus, one may wonder whether aggregating the two component indices is very useful given that there is a priori no consensus on the value that this parameter should take. Indeed, the welfare comparison of two societies based on PALE may depend on the particular value assigned to the parameter θ. Fortunately, we show that some of these comparisons do not depend on the parameter's value, *even for some*

[8] By the monotonicity of the period utility function we have $u_{NP} \geq u_P$. Assuming that being poor is no worse than being dead we have $u_P \geq 0$. Together, this implies that $\theta \in [0, 1]$.

[9] Riumallo-Herl et al. (2018) do not relate their PFLE index to a formal notion of social welfare. As our theory makes clear, the PFLE index reflects an extreme view on the trade-off between poverty and mortality, namely that being poor is as bad as being dead. One key difference between our work and Riumallo-Herl et al. (2018) is that, through our formal framework, we can derive a series of results, e.g. on the possibility to sometimes resolve "conflicting" situations, on the validity of the aggregation even when mortality is selective or on the possibility to apply this aggregation to multidimensional deprivation indices.

pairs not related by domination. That is, there exist pairs of societies for which one has larger poverty and the other larger mortality that are unambiguously ranked by PALE. Hence, the structure of expected life-cycle utility sometimes allows to extend unambiguous comparisons beyond those associated to domination.

We illustrate this graphically in Figure 1. Without aggregation, domination alone allows comparing society A with the NW quadrant (where societies have more poverty and more mortality) and the SE quadrant (where societies have less poverty and less mortality). For any value of θ, we can draw the iso-PALE$_\theta$ curves passing through A. The iso-PALE$_0$ curve (associated to $\theta = 0$) is a vertical line because this value encapsulates the view that poverty has no welfare costs. However, the iso-PALE$_1$ curve (associated to $\theta = 1$) is not a horizontal line. As a result, two additional areas for which welfare can be unambiguously compared with that of society A. This is because the iso-PALE$_\theta$ curves (associated to intermediate values of $\theta \in [0, 1]$) all lie in the area between the iso-PALE$_0$ curve and the iso-PALE$_1$ curve. The area in the NE quadrant below the iso-PALE$_1$ curve yields an unambiguously higher welfare than A. The area in the SW quadrant above the iso-PALE$_1$ yields an unambiguously lower welfare than A. The size of these new areas depends on the marginal rate of substitution of $PALE_1$ at A. If $LE(A) = 70$ and $H(A) = 20$, this marginal rate of substitution is equal to 0.011, meaning that one additional year is exactly compensated by an increase in H of 1.1% percentage points.[10]

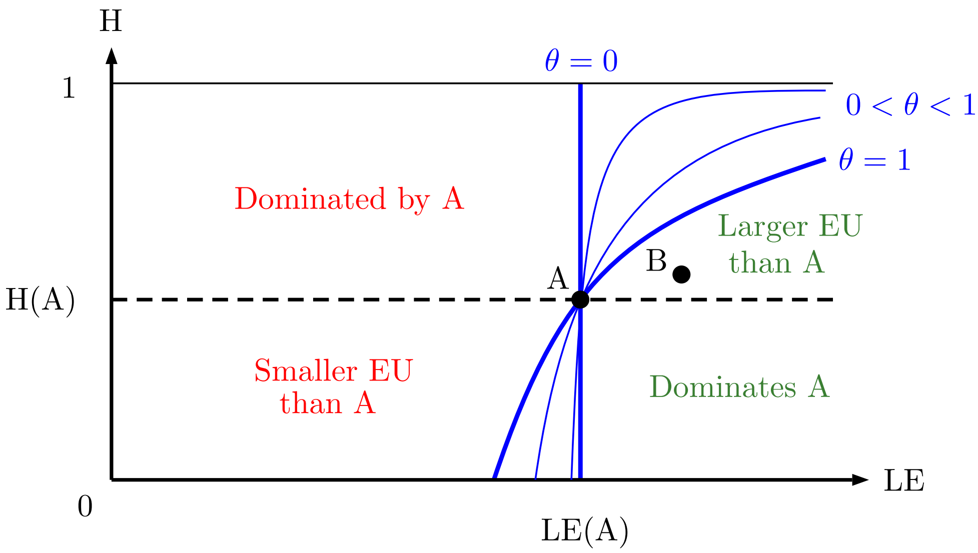


Figure 1: A simplified version of Harsanyi's expected life-cycle utility approach increases unambiguous comparisons.

We now provide some intuition for these additional unambiguous comparisons. They follow from (i) the fact that expected life-cycle utility sums period utilities and (ii) the assumption that a period in poverty is not worse than a period lost (i.e. $u_P \geq u_D$). For simplicity let us compare the life-cycle utility of two individuals i^A and i^B, who respectively live in societies A and B depicted in Figure 1. Assume that the larger poverty and smaller mortality of society B is such that the life of i^B has more periods in poverty than that of i^A, and the life of i^B also has more periods out of poverty than that of i^A. As both types of period are positively valued (ii), the value

[10]For society A and $PALE_1$, this marginal rate of substitution is given by $\frac{LE(A)(1-H(A))}{(LE(A))^2}$.

selected for the weight does not matter anymore. Indeed, i^B has a larger expected life-cycle utility because her life has more periods in each consumption status than the life of i^A.

Proposition 1 provides the conditions under which we can unambiguously compare two societies with PALE, even when they are not related by domination. The comparison is unambiguous as long as the PALE indices agree for $\theta = 0$ and $\theta = 1$.

Proposition 1 (Unambiguous comparisons of welfare)**.**
For any two societies A and B, we have $PALE_\theta(A) < PALE_\theta(B)$ *for all* $\theta \in [0, 1]$ *if and only if* $LE(A) < LE(B)$ *and* $PALE_1(A) < PALE_1(B)$*.*

There exist societies A and B with $H(A) < H(B)$ *such that* $PALE_\theta(A) < PALE_\theta(B)$ *for all* $\theta \in [0, 1]$*.*

Proof. See Appendix 5.1 for the straightforward proof. □

2.2 Applications of PALE

The data on population and mortality by country, age group and year comes from the Global Burden of Disease database (2015 version of the data). Comparable information across countries and over time is available for the 1990-2015 period and is, to our knowledge, the most comprehensive mortality data available for international comparison.[11] Data on alive deprivation come from the PovcalNet website which provides internationally comparable estimates of income deprivation level. This data set is based on income and consumption data from more than 850 representative surveys carried out in 127 low- and middle-income countries between 1981 and 2015.[12] In our empirical application, we follow the World Bank's definition of extreme income deprivation, corresponding to the \$1.9 a day threshold (Ferreira et al., 2016). We merged the two databases at the year and country level. Since the Global Burden of the Disease data are only available since 1990 and the PovCalNet data until 2015, we focus on the 1990-2015 period for a total of 113 low- and middle-income countries.

We quantify the gain that the use of the PALE index provides over the use of a dashboard of two separate indicators (LE and H). To do this, we quantify the frequence of situations for which the two indicators are "in conflict" and the percentage of these "conflicted" situations that are unambiguously ranked by PALE. Recall that we assume that the maximal value for the weight θ is equal to one. This is the most conservative approach consistent with the idea that being poor is weakly preferable to being dead. Indeed, θ=1 implies that one person-year spent in poverty is considered equivalent to one person-year prematurely lost. Choosing a lower maximal value for θ, by decreasing the maximal weight that can be given to

[11] To construct this database, population and mortality data are systematically recorded across countries and time from various data sources (official vital statistics data, fertility history data as well as data sources compiling deaths from catastrophic events). These primary data are then converted into data in five years age groups, at year and country level using various interpolations and inference methods (see Global Burden of Disease Collaborative Network (2018) and University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany) (2019) for more information on the GBD data construction). Following the literature, we only consider the point estimate in the number of deaths (see also Hoyland et al. (2012) for a critique of this approach).

[12] The website address is http://iresearch.worldbank.org/PovcalNet/povOnDemand.aspx. Each country's income deprivation level in PovCalNet is computed on a three year basis, and yearly data are obtained by linear interpolation. A complete description of the data set is given in Chen and Ravallion (2013).

the poverty component, would simply increase the number of situations that we can unambiguously compare with PALE.

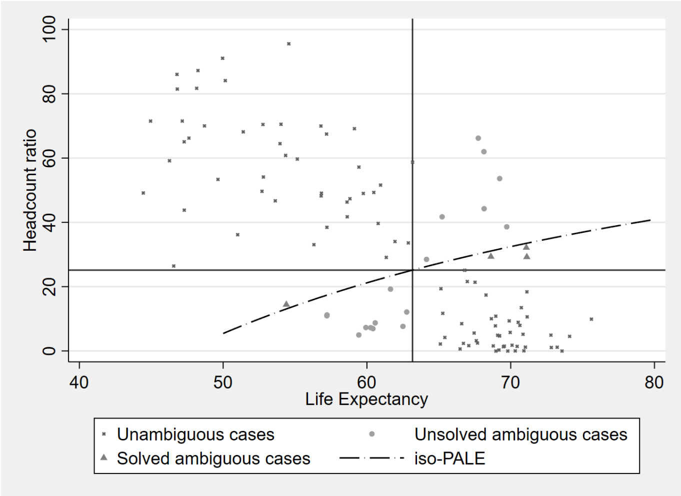
We first present an empirical version of Figure 1 above by comparing the situations of different countries in 1990 and 2015. The resulting diagram is presented in Figure 2 below. The point of reference (point A in Figure 1) chosen for this diagram is defined as a hypothetical reference country with a median head count ratio and a median life-expectancy at birth, which corresponds roughly to the situation of South Africa in 1990 an Nepal in 2015. Based on the maximal value θ equal to one, the iso-PALE_1 curve is represented by the dotted curve. All countries below this iso-PALE_1 curve have a larger PALE_1 value than the reference country. Among these, some countries, located in the north west quadrant, are obviously better off, with a larger life-expectancy and lower poverty levels. Others, located in the south east quadrant, are unambiguously worse off. In the other two quadrants, there are a significant number of countries for which the evolution of life-expectancy and poverty go in opposite directions. Among these, those represented by shaded triangles correspond to situations in which the comparison by PALE_θ is unambiguous. In the north-east quadrant, these shaded triangles signal that PALE_θ is unambiguously larger, as the higher poverty is compensated for by the lower mortality. In the south-west quadrant, these shaded triangles signal that PALE_θ is unambiguously smaller, as the lower poverty cannot compensate for the higher mortality. Countries represented by a small dots are countries we cannot rank (as the comparison depends on the chosen value given to θ).

Figure 3 replicates this exercice by comparing all pairs of countries for each year between 1990 and 2015, and reports, among all these comparisons, the proportion of cases which are ambiguous, and the share of these ambiguous cases for which PALE provides a unambiguous answer. Out of an average of 21 percent of ambiguous comparisons, PALE is able to solve one third of them. Note how the share of solved comparisons is increasing over time.

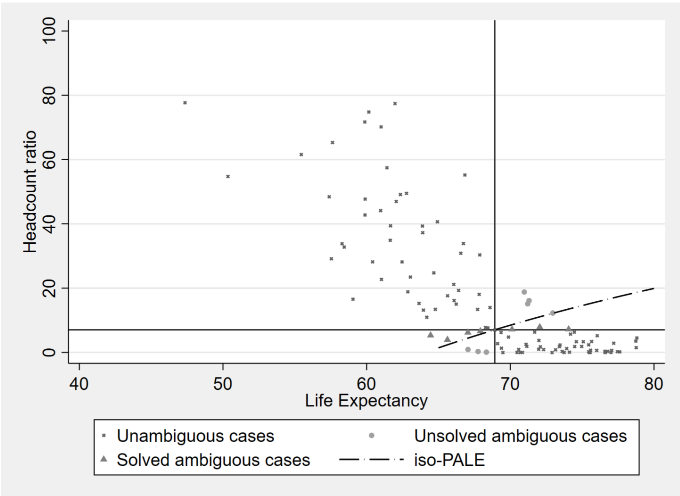
In the next figure 4, we provide PALE comparisons between present and past situations within countries. More precisely, for each year, we compare the situation in period t to the situation prevailing in the same country five years earlier. In order to represent graphically the situations for which PALE provides an unambiguous answer, we again need to represent the curve for which PALE_1 stays constant over time. By definition, PALE_1 = LE (1-HC), and thus PALE_1 increases if and only if dLE/LE > d(1-HC)/(1-HC). This simple expression allows us to contruct a figure in the (dLE/LE, d(1-HC)/(1-HC)) plan, in which the rate of growth of LE is measured on the horizontal axis, and that of (1-HC), which we refer to as the "percentage of non-poor" growth, on the vertical axis.[13] The expression above defines a "zero-growth PALE_1" curve, which represents all the combinations of the two rates of growth such that PALE_1 remains unchanged. Above this curve, PALE_1 increases, below this curve PALE_1 decreases. A priori, in a simple approach, we cannot judge the situations located in the north west and in the south east quadrants as the two indicators move in opposite directions. In these quadrants, there are two regions, one in the triangle below the curve in the north west quadrant, and one in the triangle

[13] For the sake of the graphical presentation, we excluded from the graph measures that could be considered as outliers (growth in non poverty headcount larger or smaller than 100 percent, and growth rates in life-expectancy larger than 90 percent or smaller than -40 percent). These are however adequately accounted for in the following graph.

Figure 2: Resolution of comparisons to a median country in 1990 and 2015



(a) 1990



(b) 2015

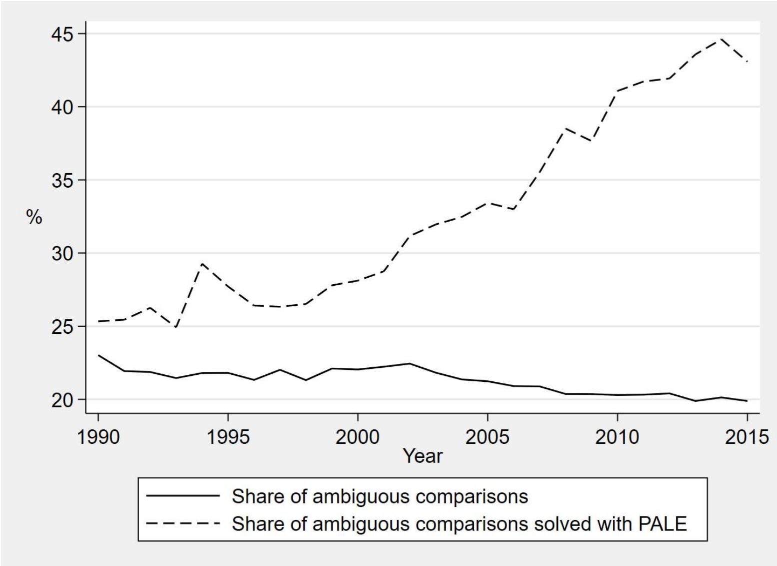
Only observations with life-expectancy > 40 are reported for readibility.

above the curve in the south east quadrant for which PALE is able to provide a clear welfare comparison. These areas are the ones with shaded triangles, which represent situations in which either welfare unambiguously improved (in the south east quadrant) or unambiguously decreased (in the north west quadrant). If we agree with the assumption that being dead is worse than being poor, these points represent situations in which, in a particular country, the situation either strictly improved or deteriorated compared to the situation in the same country five years earlier.[14]

Finally, Figure 5 reports, using the same comparisons, the evolution over time of the frequency of ambiguous situations in which life-expectancy and poverty moved in opposite directions in one country between t and t+5, and the share of these ambiguous situations for which the most conservative definition of PALE provides a

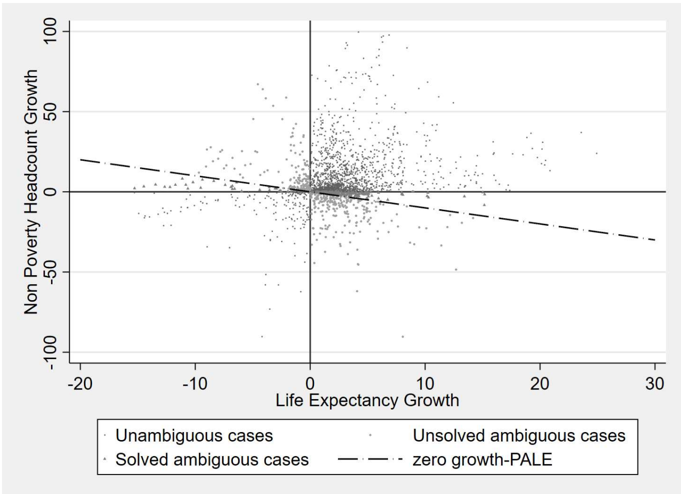
[14]Again, if being dead is strictly worse than being poor, so that θ is always strictly lower than one, more situations can be strictly signed for all other values of θ. They are located in the triangle above the "zero-growth $PALE_1$" in the NW quadrant, and in the triangle below the "zero-growth $PALE_1$" in the SE quadrant.

Figure 3: Evolution of the resolution of ambiguous inter-country comparisons, 1990-2015



Reading: in 1990, countries had on average 23% of ambiguous comparisons, out of which 25% where solved on average by the use of PALE.

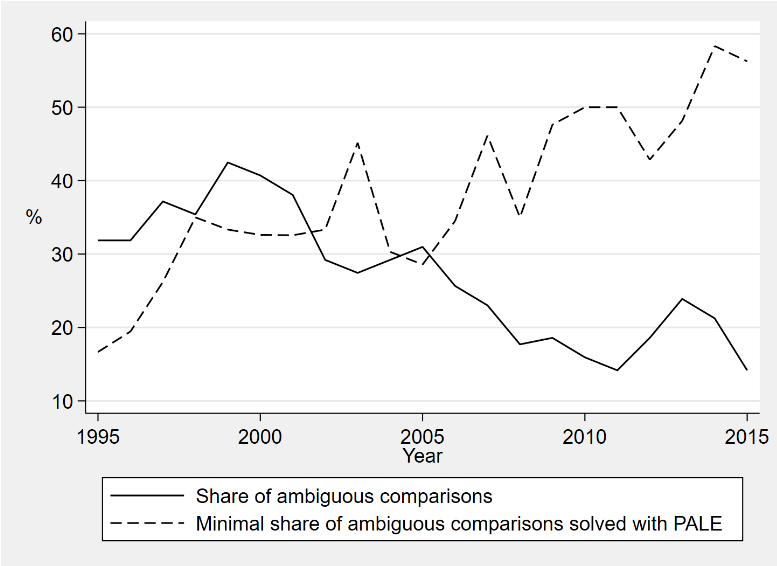
Figure 4: Resolution of ambiguous countries' evolutions, 1990-2015



Only observations for which the non poverty growth is lower than 100% and for which the growth of life-expectancy is between 90% and - 40% are represented for readibility.

clear ranking. Overall the share of ambiguous comparisons declines from about 30 to 20 percent over the period considered (with an overall average of 27 percent). Out of these, we can solve an increasingly high share of welfare comparisons from about 20 in 1990 to more than 50 percent in the last years considered. Across all years, PALE is able to solve an average of 38 percent of these comparisons.

Figure 5: Evolution of the resolution of ambiguous countries' evolutions, 1990-2015



2.3 PALE beyond the independence case

We have shown in Section 2.1 that PALE corresponds to a simplified version of expected life-cycle utility (Eq. 3), under the assumption of independence. However, independence is unlikely in practice: mortality is selective, i.e. the poor die younger than the non-poor (Chetty et al., 2016). Canudas-Romo (2018) points to this limitation when criticizing the PFLE index of Riumallo-Herl et al. (2018), which corresponds to $PALE_1$. This may cast some doubts on whether PALE is a valid measure of welfare. In this section, we show that this is indeed the case. Under our two assumptions, PALE is a normalized expression of the expected life-cycle utility in any stationary society.

There is a discrete set of periods $\{\dots, t-1, t, t+1, \dots\}$. In each period, some individuals are born, all alive individuals are assigned a consumption status for the period (P or NP) and some individuals die (at the end of the period). We define the life of an individual i as the list of consumption statuses $l_i = (l_{i0}, \dots, l_{id})$ she enjoys between age 0 and age $d \in \{0, \dots, a^* - 1\}$ at which she dies, where $l_{ia} \in \{NP, P\}$. The set of lives is thus $L = \cup_{d \in \{0,\dots,a^*-1\}} \{NP, P\}^{d+1}$.

The number of newborns in period t is denoted by n_t. The profile of lives for the cohort born in t is denoted by $C_t = (l_i)_{i \in \{1,\dots,n_t\}}$, where $\{1, \dots, n_t\}$ is the set of newborns in t. Clearly, the profile of lives C_t contains all the information necessary to compute a newborn's expected life-cycle utility (Eq. (3)). Let $n_t(a)$ denote the number of individuals born in period t who are still alive when reaching age a. In particular, we have $n_t(0) = n_t$. Let $p_t(a)$ denote the number of individuals born in period t who are poor when at age a. Thus, we have $p_t(a) \leq n_t(a)$. By definition, the probability that an individual born in t survives to age a is $S_t(a) = \frac{n_t(a)}{n_t}$, and the conditional probability that an individual born in t will be poor when reaching age a is $\pi_t(a) = \frac{p_t(a)}{n_t(a)}$. To compute Eq. (3), it is sufficient to know the distribution on the set of lives that C_t implicitly defines. We denote this distribution by $\Gamma_t : L \to [0, 1]$, with $\sum_{l \in L} \Gamma_t(l) = 1$.

In period t, we cannot observe the profile of lives for the cohort born in t. The only elements of C_t that we observe in that period are $n_t(0)$, $p_t(0)$ and $n_t(1)$. However, we also have information about the profile of lives of the cohorts born period *before* t. Formally, let a society S_t be the list of profiles of lives for all cohorts born during the a^* periods in $\{t-(a^*-1),\dots,t\}$, i.e. $S_t = (C_{t-a^*+1},\dots,C_t)$. In period t, we observe the number N_t of individuals who are alive in t, i.e.

$$N_t = \sum_{a=0}^{a^*-1} n_{t-a}(a),$$

the fraction H_t of alive individuals who are poor in t,

$$H_t = \frac{\sum_{a=0}^{a^*-1} p_{t-a}(a)}{\sum_{a=0}^{a^*-1} n_{t-a}(a)},$$

and the age-specific mortality vector $\mu^t = (\mu_0^t,\dots,\mu_{a*-1}^t)$ in period t where for each $a \in \{0,\dots,a^*-1\}$ we have

$$\mu_a^t = \frac{n_{t-a}(a) - n_{t-a}(a+1)}{n_{t-a}(a)},$$

with $\mu_{a^*-1}^t = 1$.

We now show that the information available in period t is sufficient to infer the value of Eq. (3) in stationary societies. The particularity of stationary societies is to have their natality, mortality and poverty constant over time. In a stationary society, all the outcomes observed in a period simply replicates in the next period. More formally, a society is stationary if both the distribution of lives and the size of generations are constant over the last a^* periods.

Definition 1 (Stationary society)**.**
A society S_t is stationary if for any period $t' \in \{t-a^*+1,\dots,t\}$ we have

- $\Gamma_{t'} = \Gamma_t$ *(constant distribution of lives),*
- $n_{t'} = n_t$ *(constant size of cohorts).*

It follows immediatly from this definition that $n_t(a) = n_{t-a}(a)$ and $p_t(a) = p_{t-a}(a)$ for all $a \in \{1,\dots,a^*-1\}$.[15] These equalities lead to the following Lemma, which allows us to relate Eq. (3) to the information available in period t.[16]

Lemma 1. *If S_t is stationary,*

$$S_t(a) = \Pi_{k=0}^{a-1}(1-\mu_k^t) \qquad \text{for all } a \in \{0,\dots,a^*-1\}, \tag{5}$$

$$N_t = n_t * LE_t, \tag{6}$$

$$N_t * H_t = n_t * \sum_{a=0}^{a^*-1} S(a)\pi(a). \tag{7}$$

Proof. See Appendix 5.2 □

[15] Clearly, a constant distribution of lives is not sufficient for these equalities, one also needs a constant size of cohorts.

[16] Lemma 1 also requires that $n_t(a+1) = n_{t-a}(a+1)$ for all $a \in \{0,\dots,a^*-2\}$, which follows from the definition of a stationary society.

The three equations in Lemma 1 imply that an individual who is born in a stationary society can infer her expected life-cycle utility from the information available at the year of her birth. (These direct relationships between current and future outcomes in stationary societies are well-known to demographers (Preston et al., 2000).) We illustrate this important insight using an example. Consider a stationary society for which two individuals are born in each cohort, one living only for one period in poverty and the other living for two periods out of poverty, i.e. $n = 2$, $l_1 = (P)$ and $l_2 = (NP, NP)$. In period t, three individuals are alive: the poor born in t, the non-poor born in t and the non-poor born in $t-1$. Also, two individuals die at the end of period t: the poor born in t and the non-poor born in $t-1$. Eq. (5) states that the mortality rates observed in period t (the right hand side of the equation) can be used to infer the mortality rates that the newborn can expect to face during her life-cycle (the left hand side). Thus, in our example, a newborn observes that, at the end of period t, half of the individuals of age 0 die and all individuals of age 1 die. Eq. (5) implies that a newborn has a 50 percent chance to survive period t and a zero percent chance to survive period $t+1$. According to Eq. (6), the number of individuals who are alive in period t, N_t, is equal to the number of person-periods in the profile of lives of the cohort born in period t. In our example, there are three individuals alive in period t and there are three person-periods in $C_t = (l_1, l_2) = (P; NP, NP)$. Finally, Eq. (7) states that the number of poor observed in period t, $N_t * H_t$, is equal to the number of person-periods of poverty in the profile of lives of the cohort born in period t. Indeed, there is one poor individuals alive in period t and one person-period P in C_t.

Lemma 1 shows that, in a stationarity society, the poverty and mortality observed in a given period perfectly define the profile of lives of newborns. Proposition 2 shows that PALE is a normalisation of the expected life-cycle utility of a newborn in a stationary society even when mortality is selective, i.e. when the conditional probability of being poor depends on age.

Proposition 2 (Connection between Harsanyi and PALE)**.**
If society S_t is stationary, then $PALE_\theta = \frac{EU_t}{u_{NP}}$.

Proof. The result follows directly when substituting Eq. (6) and (7) into Eq. (3). □

We now discuss the implications of this result. The result shows that a newborn can interpret PALE as a normalization of her expected life-cycle utility if she assumes that she is born in a stationary population, i.e. if she assumes that poverty and mortality that she observes at the time of her birth remain unchanged over the course of her life. Clearly, in practice, populations are not stationary and we cannot in general interpret PALE as the expected life-cycle utility of a newborn. Indeed, the poverty and mortality observed at birth might not be good predictors for the future, in particular if mortality and mortality decline over time with medical progress or economic growth. Therefore, PALE should *not* be understood as a projection or a forecast of the expected life-cycle utility.

This being said, the validity of PALE as an indicator of a society's welfare in period t does not rely on the interpretation one can give to PALE. More fundamentally, this validity depends essentially on whether PALE aggregates the welfare losses due to the poverty and mortality observed in t in a meaningful way. Our result shows that this aggregation is meaningful in the sense that it is directly related to the expected life-cycle utility of an individual confronted throughout her life to the poverty and mortality observed in t. This is perfectly consistent with the idea of evaluating welfare in period t. Indeed, the aggregation of the welfare losses observed in period t should not depend on the future evolutions of poverty and mortality. For instance, a transitory mortality or poverty shock – due to war or to another disaster – does reduce current welfare, even if the country fully recovers in the next period. In contrast, the transitory nature of the shock implies that its consequences are spread across all the current generations. Its impact on the actual expected life-cycle utility of newborns can be therefore be negligible, or nil if the shock did not affect the mortality rates of newborns.

Observe that the same point can be made about life-expectancy at birth (LE). In practice, this index is computed from the mortality vector observed in the period (from Eq. (5)). As a result, this index typically does not correspond to the average lifespan in a cohort born in a society that is not stationary. However, the aggregation that this index makes of the mortality observed in the period is still considered valid and is widely interpreted as the number of years of life that a newborn expects to live in a society.

3 A transparent index of deprivation

One limitation of PALE is that it does not reflect the distributional concern in both its dimensions. Although PALE does focus on low quality of life due to poverty, PALE does not focus on low quantity of life. Indeed, an additional year of life bestowed to an individual dying in old age has the same impact on PALE as an additional year of life given to an individual dying in young age. Clearly, lifespan is distributed less unequally than consumption (Peltzman, 2009), which slightly tunes down the need to capture unequal lifespans when monitoring human development. Nevertheless, concerns around unequal lifespans justify the use of an indicator that is sensitive to very low lifespans.

For this reason, we apply our welfare index to measuring deprivation in the quality and quantity of life. Multidimensional poverty indices capturing the quality and quantity of life are plagued by the same limitations that we emphasized for welfare indices. They typically lack solid theorethical foundationsor are black boxes with opaque trade-offs (Ravallion, 2011b).

Two properties of a measure of deprivation require us to adapt the PALE index. First, deprivation is the opposite concept of welfare, i.e. deprivation decreases when welfare increases. Second, we must define deprivation in the quantity of life. Borrowing from a long tradition focussing on absolute poverty, we consider as deprived an individual who dies prematurely, i.e. who dies before reaching a minimal age threshold $\hat{a}$. Following Baland et al. (2021), we call "lifespan deprived" an individual who dies before reaching the age threshold. As we show in the next section, under these two properties, PALE naturally leads to a particular index of deprivation.

3.1 The ED index

We call expected deprivation at birth (ED) the index defined by PALE in the deprivation context. ED is based on an indicator of mortality different from LE. Indeed, when focusing on deprivation in the quantity of life, only the years of life lost *before* reaching the minimal age threshold $\hat{a}$ matter. We therefore define another indicator of mortality, the lifespan gap expectancy, which measures the number of years that a newborn expects to lose prematurely.[17] As above, letting $n_t(a)$ denote the number of individuals born in period t who survive at least to age a and $n_t = n_t(0)$, we have[18]:

$$LGE_{\hat{a}}(C_t) = \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \frac{n_t(a) - n_t(a+1)}{n_t}.$$

We illustrate in Figure 6 the close connections between $LGE_{\hat{a}}$ and LE, where $LE = \sum_{a=0}^{a^*-1} \frac{n_t(a)}{n_t}$. In the figure, we construct a counterfactual population pyramid by reporting for each age a the number $n_t(a)$ of newborns who are still alive at age a. As explained in Section 2.3, this counterfactual pyramid exactly corresponds to the population pyramid in period t if the society is stationary in period t.[19] In the left panel of Figure 6, LE is proportional to the area below the counterfactual "stationary" population pyramid. By contrast, $LGE_{\hat{a}}$ is proportional to the area between this "stationary" population pyramid and the age threshold. In the right panel, we illustrate that, for large enough age thresholds, $LGE_{\hat{a}}$ is the complement of LE. Formally, when $\hat{a} \geq a^*$ we have $LGE_{\hat{a}} = \hat{a} - LE$.

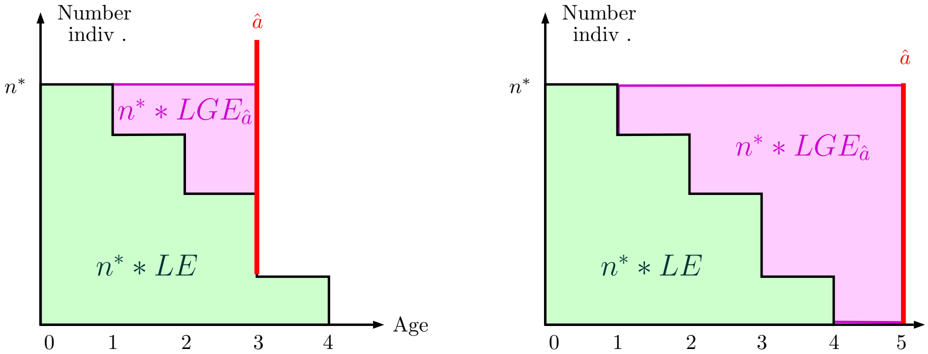


Figure 6: In the Left panel, the light grey area below the counterfactual "stationary" population pyramid is a multiple of LE and the dark grey area is a multiple of $LGE_{\hat{a}}$. The Right panel illustrates that, when $\hat{a} \geq a^*$ we have $LGE_{\hat{a}} = \hat{a} - LE$.

The expected deprivation index, ED, aggregates the poverty and lifespan deprivation expected by a newborn, if she considers facing, throughout her life-cycle, the poverty and mortality prevailing at the time of her birth. It combines a component for deprivation in the quality of life and a component for deprivation in the quantity

[17] $LGE_{\hat{a}}$ is a particular version of the Years of Potential Life Lost, an indicator used in medical research in order to quantify and compare the burden on society due to different death causes (Gardner and Sanborn, 1990).

[18] See Proposition 5 for a mathematical expression for $LGE_{\hat{a}}$ that only depends on the poverty and mortality observed in period t.

[19] In a stationary society, the current population pyramid can be obtained by successively applying the current age-specific mortality rates to each age group.

of life:

$$ED_\theta = \underbrace{\frac{LGE_{\hat{a}}}{LE + LGE_{\hat{a}}}}_{quantity\ deprivation} + \theta \underbrace{\frac{LE * H}{LE + LGE_{\hat{a}}}}_{quality\ deprivation}, \tag{8}$$

where the parameter $\theta \in [0, 1]$ is defined in exactly the same way as for PALE.

Both components have the same denominator, which measures a normative lifespan corresponding to the sum of LE and $LGE_{\hat{a}}$. This normative lifespan can be interpreted as the (counterfactual) life-expectancy at birth that would prevail if all premature deaths were postoned to the age threshold. It is at least as large as LE, and would be equal to LE if the age treshold is equal to 0.

The numerator of each term measures the expected number of years characterized by one of the two dimensions of deprivation. The numerator of the quantity deprivation component measures the number of years that a newborn expects to loose prematurely (when observing mortality in the period) given the age threshold, $\hat{a}$. The numerator of the quality deprivation component measures the number of years that a newborn expects to spend in poverty (when observing the poverty and mortality in the period). As said above, these expectations are correct in the independence case (see Section 2.1) or in stationary societies (see Section 2.3). In a stationary population, the term $LE * H$ can be interpreted as the number of years that a newborn expects to spend in poverty, because poverty and mortality observed in the period perfectly reflect the poverty and mortality her cohort will be confronted to in the future. Of course, this interpretation requires that the society is stationary (or that the two dimensions are independent), which imply some caution when interpreting ED. However, this does not invalidate its use as an indicator of deprivation in the current period (see Section 2.3): again, a widely used index such as life-expectancy suffers from exactly that same limitation but is nevertheless interpreted as if in a stationary society.

Finally, the definition of ED_θ is such that each year prematurely lost is as bad as $1/\theta$ years spent in poverty. This trade-off between the relative costs of poverty and mortality is the same as for $PALE_\theta$.[20] When $\theta = 1$, ED has a transparent interpretation, as it computes the expected proportion of the normative lifespan that a newborn expects to lose prematurely or spend in alive deprivation (assuming a stationary society). Note also that, when lifespan deprivation is absent ($LGE = 0$), ED_1 corresponds to the headcount ratio, H.

In Proposition 3, we establish the relation between ED and PALE under the two properties of deprivation discussed above. First, ED_θ leads to the opposite ranking as $PALE_\theta$ as long that the age threshold is irrelevant, i.e. larger than the maximal age. When the age treshold is binding (smaller than the maximal age), the rankings obtained under ED_θ do not correspond to the opposite of the ranking obtained under $PALE_\theta$.

Proposition 3 (Connection between ED and PALE)**.**
If $\hat{a} \geq a^*$, then $PALE_\theta = \hat{a}(1 - ED_\theta)$, which implies that, for any two societies A

[20]We assume here that the welfare costs of a year prematurely lost is u_{NP}.

and B,

$$PALE_\theta(A) \geq PALE_\theta(B) \Leftrightarrow ED_\theta(A) \leq ED_\theta(B).$$

Proof. See Appendix 5.3. □

Taken together, Propositions 2 and 3 show that ED aggregates two indices of mortality, LE and $LGE_{\hat{a}}$, with an index of poverty, H, in a way which is consistent with life-cycle preferences. This improves on standard multidimensional poverty indices, but ED also relies on a normative parameter that weights the two dimensions. Again, one may wonder whether aggregating the two component indices is very useful when there is no consensus on the value that this parameter should take. Proposition 4 shows that some pairwise comparisons do not depend on the latter value, *even if these pairs are not related by domination.*

Proposition 4 (Unambiguous comparisons of deprivation)**.**
For any $\hat{a} \in \{2, \ldots, a^*\}$ and any two societies A and B, $ED_\theta(A) > ED_\theta(B)$ for all $\theta \in [0, 1]$ if and only if $ED_0(A) > ED_0(B)$ and $ED_1(A) > ED_1(B)$.

There exist societies A and B with $H(A) < H(B)$ such that $ED_\theta(A) > ED_\theta(B)$ for all $\theta \in [0, 1]$.

Proof. See Appendix 5.4. □

3.2 Relation with other indices of deprivation

We limit the comparisons to other indices in the literature to a comparison with the index of generated deprivation (GD) proposed by Baland et al. (2021). The reason is that GD index is closest to the ED index, and Baland et al. (2021) discuss in details the relationships between GD and other indices of multidimensional poverty. In a nutshell, ED and GD are identical in stationary societies, but ED has a simpler interpretation than GD, reacts faster to permanent mortality shocks and is slightly less demanding in terms of data. However, GD is decomposable in subgroup whereas ED is not.

The GD index is defined as[21]

$$GD_\theta = \underbrace{\frac{YL_t}{N_t + YL_t}}_{quantity\ deprivation} + \theta \underbrace{\frac{N_t * H_t}{N_t + YL_t}}_{quality\ deprivation}, \tag{9}$$

where $\theta \in [0, 1]$, $N_t = \sum_{a=0}^{a^*-1} n_{t-a}(a)$ is the population observed in t and YL_t is the number of years of life prematurely lost due to mortality in year t, i.e.

$$YL_t = \sum_{a=0}^{\hat{a}-2} n_{t-a}(a) * \mu_a^t * (\hat{a} - (a+1)),$$

with $\mu^t = (\mu_0^t, \ldots, \mu_{a^*-1}^t)$ again standing for the vector of age-specific mortality rates.

Mathematically, the GD index is based on the same two components, one capturing quality deprivation and the other quantity deprivation. Moreover, the same

[21] The generated deprivation index defined in Baland et al. (2021) is $\frac{1}{\theta} GD_\theta$, which is ordinally equivalent to GD_θ since θ is a constant.

normative weights is used for these two components. One disadvantage of GD is that these components are harder to interpret. This is because GD combines a number of poor with a number of years of life prematurely lost. The fundamental reason for this is that the number of poor in a particular year corresponds to the number of years lived in poverty in that year. This equivalence also explains why the denominator of GD sums a number of individuals with a number of years. By contrast, the numerators of the two terms in ED are more easily interpretable. They are the number of years that a newborn expects to prematurely loose or spend in poverty, if she expects mortality and poverty to stay at their observed levels.

The following proposition establishes that GD and ED are identical in stationary societies.

Proposition 5 (ED and GD are identical in stationary societies)**.**
If society S_t is stationary,

$$LGE_{\hat{a}}(S_t) = \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \mu_a^t * \prod_{k=0}^{a-1} (1 - \mu_k^t), \tag{10}$$

which yields $GD_\theta(S_t) = ED_\theta(S_t)$ for all $\theta \in [0, 1]$.

Proof. See Appendix 5.5. □

Following Proposition 5, ED and GD yield the same ranking for stationary societies. However, societies are typically not stationary and ED and GD rank countries in different ways.

The main difference between ED and GD comes from the way the two indices compute the number of years prematurely lost. ED takes the perspective of a newborn who faces throughout her life the mortality rates observed in t. In contrast, GD computes the number of years that are lost by the current population due to the premature mortality observed in t. It records, over all premature deaths in $t*$, the number of years prematurely lost. Thus, if an individual dies at age 20 and the age threshold is 70, her premature death leads to a loss of 50 years of life. It is worth noting here that ED also counts the number of years prematurely lost, but instead of being computed for the actual population pyramid, ED uses a counterfactual population pyramid, which is the one that would prevail in a stationary society characterized by the age-specific mortality rates observed in t.

A major implication of this difference is that ED is more reactive to policy changes than GD. Consider a permanent mortality shock. The population dynamics is such that a transition phase sets in during which the population pyramid slowly adjusts to the new mortality rates. This transition stops when a new stationary population pyramid is reached, typically after a^* periods. GD records each step of this transition and therefore exhibits inertia in its response to a permanent mortality shock. By contrast, ED immediately refers to the new stationary population pyramid and disregard the inertia caused by transitory demographic adjustments.

We illustrate this difference between ED and GD with the help of a simple example. Consider a population with a fixed natality $n_t(0) = 2$ for all periods t. At each period, all alive individuals are non-poor, implying that $H_t = 0$. For all $t < 0$, we assume a constant mortality vector $\mu^t = \mu^* = (0, 1, 1, 1)$, so that each individual lives exactly two periods. Let us assume $\hat{a} = 4$, so that an individual dies prematurely if she dies before her fourth period of life. Before period $t = 0$, the population pyramid is stationary, and the two indices are equal to $1/2$ because there is no poor and individuals live for two periods instead of four. Consider now a permanent shock starting from period 0 onwards, such that half of the newborns die after their first period of life: $\mu^0 = (1/2, 1, 1, 1)$. The population pyramid returns to its stationary state in period 1, after a (mechanical) transition in period 0. This example is illustrated in Figure 7.

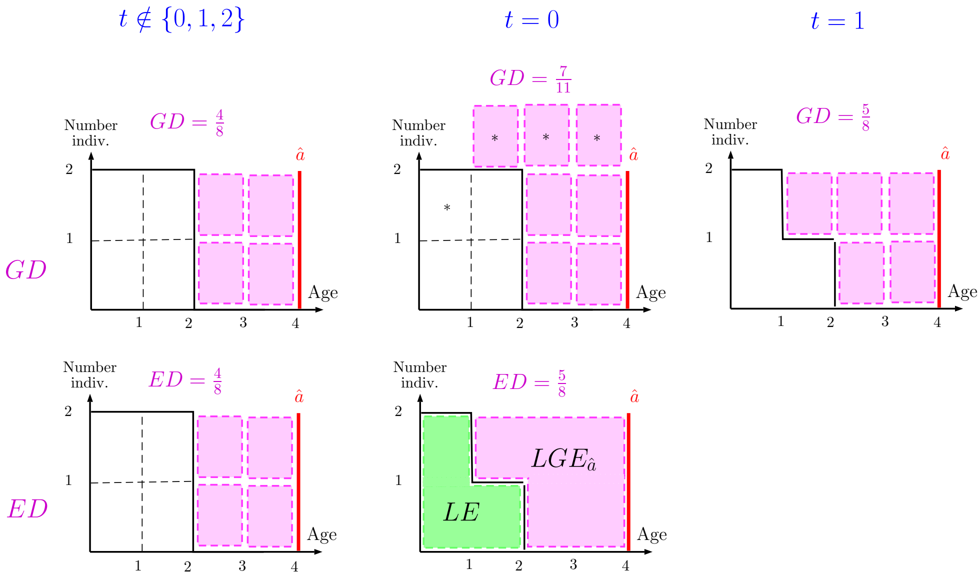


Figure 7: Response of GD and ED to a permanent mortality shock in $t = 0$. The years prematurely lost are shaded.

Consider first GD. In period 0, the actual population pyramid is not stationary because of the mortality shock. The premature death of one newborn leads to the loss of three years of life. Also, two one-year old individuals die in period 0, each losing two years of life. There are thus 7 years of life prematurely lost in period 0, and GD takes value $7/11$. In period 1, the population pyramid is stationary, and GD is equal to $5/8$ from then on.

We now turn to ED. Even if the actual population pyramid is not stationary in period 0, ED is immediately equal to $5/8$ since it records premature mortality as if the population pyramid had already reached its new stationary level. ED focusses on the newborn and the one-year old who die prematurely, ignoring that there are two one-year old dying in the actual population pyramid in period 0 (which is a legacy of the past).

Baland et al. (2021) show that GD is essentially the only index decomposable in subgroups to compare stationary societies in a way that satisfy basic properties.[22] As a result, ED cannot be decomposable in subgroups. This is no surprise given that ED is based on LE, which cannot be decomposed in subgroups. In Appendix 6, we show that ED is essentially the only index that is independent on the actual population pyramid and compares stationary population in a way that respect basic properties of deprivation. This is important since it implies that information on the

[22] Baland et al. (2021) show that basic properties define an Inherited Deprivation index, ID, which is based on premature mortality that took place in the past. They then show that GD is the only index that is based on premature mortality in year t, which is equivalent to ID in stationary societies and decomposable in subgroups.

actual population pyramid is irrelevant for ED, since only age-specific mortality rates are required.

4 Concluding remarks

An important limitation of indices PALE and ED is that they account for the distributional concern “dimension-by-dimension” instead of accounting for them in terms of life-cycle utility. Indeed, our indices are insensitive to the allocation of years of life prematurely lost between the poor and the non-poor. This allocation may however have important implications for the distribution of life-cycle utility. Indeed, when the poor die early, they combine low achievements in the two dimensions and the difference between their life-cycle utility and that of the non-poor increases. Without denying the importance of this limitation, let us first note that this limitation is shared by most standard indices of human development.[23] Second, addressing this limitation requires data that are typically not available. One natural way of accounting for such “concentration” of deprivations on the same individuals would be to define as “life-cycle poor” individuals whose life-cycle utility is smaller than that of a reference life l^*. Then, an index of human development could for instance correspond to the expected fraction of newborns who will be “life-cycle poor”. This type of index would not be ad-hoc, but would require better data, combining poverty and mortality at the individual level, than what is currently available in most countries. Moreover, this type of data, recording mortality up to a given threshold, would necessarily be historical in nature, with little relevance to the current situation. Alternatively, one may want to define indices that are less demanding in terms of information, and would be based on the observed mobility between poverty and non-poverty, as well as mortality figures for the poor and the non-poor. Some additional assumptions would then be needed to translate this information into live profile for newborns.

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[23] To the best of our knowledge, only the global MPI accounts, in an indirect way, for such concentration. In a nutshell, the deprivation-score of an individual is increased if she lives in a household that has experienced the death of a less than 18 year child in the past five years. Arguably, this aggregation of the quantity and quality of life is essenitally practical, as it is not related to any definition of life-cycle utility, it depends on the definition of the household and it does not account for multiple deaths in the same households.

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5 Appendix 1: Proofs

5.1 Proof of Proposition 1

We start with the "only if" part of the first statement. Assume to the contrary that $LE(A) > LE(B)$ or $PALE_1(A) > PALE_1(B)$. As $PALE_0 = LE$, this implies that $PALE_\theta(A) > PALE_\theta(B)$ for some $\theta \in \{0,1\}$ and therefore we cannot have $PALE_\theta(A) < PALE_\theta(B)$ for all $\theta \in [0,1]$.

We turn to the "if" part of the first statement. By definition of the PALE index, we have to show that

$$LE(B) - LE(A) > \theta * (H(B) - H(A)), \tag{11}$$

for all $\theta \in [0,1]$. As $LE(A) < LE(B)$, we have $LE(B)-LE(A) > 0$. As $PALE_1(A) < PALE_1(B)$, we have $LE(B)-LE(A) > (H(B)-H(A))$. It immediately follows that the inequality (11) is verified for all values of θ smaller than 1.

From the first statement, proving the second statement only requires providing A and B with $H(A) < H(B)$ such that $LE(A) < LE(B)$ and $PALE_1(A) < PALE_1(B)$. If $H(A) = 0.2$, $H(B) = 0.4$, $LE(A) = 50$ and $LE(B) = 75$ we have $PALE_1(A) = 40$ and $PALE_1(A) = 45$, the desired result.

5.2 Proof of Lemma 1

We first prove Eq (5). As S_t is stationary, we have $n_t(k) = n_{t-k}(k)$ for all $k \in \{1,\ldots,a^*-1\}$ and $n_t(k+1) = n_{t-k}(k+1)$ for all $k \in \{0,\ldots,a^*-2\}$. Therefore, we have for all $a \in \{1,\ldots,a^*-1\}$ that

$$\begin{aligned}
S_t(a) &= \frac{n_t(a)}{n_t}, \\
&= \Pi_{k=0}^{a-1}\frac{n_t(k+1)}{n_t(k)}, \\
&= \Pi_{k=0}^{a-1}\frac{n_{t-k}(k+1)}{n_{t-k}(k)}, \\
&= \Pi_{k=0}^{a-1}(1-\mu_k^t).
\end{aligned}$$

We then prove Eq (6). As S_t is stationary, we have $n_t(a) = n_{t-a}(a)$ for all $a \in \{1,\ldots,a^*-1\}$. Recalling that $S_t(a) = \frac{n_t(a)}{n_t}$, we can successively write

$$\begin{aligned}
LE_t &= \sum_{a=0}^{a^*-1} S_t(a), \\
&= \frac{\sum_{a=0}^{a^*-1} n_t(a)}{n_t}, \\
&= \frac{\sum_{a=0}^{a^*-1} n_{t-a}(a)}{n_t}, \\
&= N_t/n_t.
\end{aligned}$$

Finally, we prove Eq. (7). As S_t is stationary, we have $p_t(a) = p_{t-a}(a)$ for all $a \in \{1,\ldots,a^*-1\}$. Recalling that $\pi_t(a) = \frac{p_t(a)}{n_t(a)}$ and $S_t(a) = \frac{n_t(a)}{n_t}$, we can successively write

$$
\begin{aligned}
H_t &= \frac{\sum_{a=0}^{a^*-1} p_{t-a}(a)}{\sum_{a=0}^{a^*-1} n_{t-a}(a)}, \\
&= \frac{\sum_{a=0}^{a^*-1} p_t(a)}{N_t}, \\
&= \frac{\sum_{a=0}^{a^*-1} \pi_t(a) S_t(a) n_t}{N_t}.
\end{aligned}
$$

5.3 Proof of Proposition 3

We show that $LE + LGE_{\hat{a}} = \hat{a}$ when $\hat{a} \geq a^*$.

$$
\begin{aligned}
LGE_{\hat{a}}(C_t) &= \sum_{a=0}^{\hat{a}-1} \hat{a} * \frac{n_t(a) - n_t(a+1)}{n_t} - \sum_{a=0}^{\hat{a}-1} (a+1) * \frac{n_t(a) - n_t(a+1)}{n_t}, \\
&= \frac{1}{n_t} \left(\hat{a} * (n_t(0) - n_t(\hat{a})) - \sum_{a=0}^{\hat{a}-1} n_t(a) + \hat{a} * n_t(\hat{a}) \right), \\
&= \hat{a} - \sum_{a=0}^{\hat{a}-1} \frac{n_t(a)}{n_t}.
\end{aligned}
$$

By definition of a^*, we have $n_t(a) = 0$ for all $a \geq a^*$. When $\hat{a} \geq a^*$, this implies that $\sum_{a=0}^{\hat{a}-1} \frac{n_t(a)}{n_t} = \sum_{a=0}^{a^*-1} \frac{n_t(a)}{n_t}$, where by definition $LE = \sum_{a=0}^{a^*-1} \frac{n_t(a)}{n_t}$, the desired result.

Thus, when $\hat{a} \geq a^*$, $PALE_\theta$ is a linear function of ED_θ that depends negatively on ED_θ. Therefore, these two indicators yields opposite ranking of all societies A and B, i.e. $PALE_\theta(A) \geq PALE_\theta(B) \Leftrightarrow ED_\theta(A) \leq ED_\theta(B)$.

5.4 Proof of Proposition 4

We start with the "only if" part of the first statement. Assume to the contrary that $ED_0(A) < ED_0(B)$ or $ED_1(A) < ED_1(B)$. This implies that $ED_\theta(A) < ED_\theta(B)$ for some $\theta \in \{0, 1\}$ and therefore we cannot have $ED_\theta(A) > ED_\theta(B)$ for all $\theta \in [0, 1]$.

We turn to the "if" part of the first statement. By definition of the ED index, we have to show that

$$
\frac{LGE_{\hat{a}}(A)}{LE(A) + LGE_{\hat{a}}(A)} - \frac{LGE_{\hat{a}}(B)}{LE(B) + LGE_{\hat{a}}(B)} > \theta \left(\frac{LE(B) * H(B)}{LE(B) + LGE_{\hat{a}}(B)} - \frac{LE(A) * H(A)}{LE(A) + LGE_{\hat{a}}(A)} \right) \tag{12}
$$

for all $\theta \in [0, 1]$. As $ED_1(A) > ED_1(B)$, Eq. (12) holds for $\theta = 1$. As $ED_0(A) > ED_0(B)$, the LHS of Eq. (12) is strictly positive. It immediately follows that the inequality (12) is verified for all values of θ smaller than 1.

From the first statement, proving the second statement only requires providing A and B with $H(A) < H(B)$ such that $ED_0(A) > ED_0(B)$ and $ED_1(A) > ED_1(B)$. If $H(A) = 0.5$, $H(B) = 0.6$, $LE(A) = 1$, $LE(B) = \hat{a}$, $LGE_{\hat{a}}(A) = \hat{a} - 1$ and $LGE_{\hat{a}}(B) = 0$ we have $ED_0(A) = \frac{\hat{a}-1}{\hat{a}}$, $ED_0(B) = 0$, $ED_1(A) = \frac{\hat{a}-0.5}{\hat{a}}$, $ED_1(B) = H(B)$. This yields the result since $\hat{a} \geq 2$ implies $ED_0(A) > 0$ and $ED_1(A) \geq 0.75$.

5.5 Proof of Proposition 5

We first prove Eq. (10). As S_t is stationary, we have that $n_t(a) = n_{t-a}(a)$ and $n_t(a+1) = n_{t-a}(a+1)$ for all $a \in \{0, \ldots, a^* - 1\}$. We can thus successively write

$$\begin{aligned} LGE_{\hat{a}}(S_t) &= \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \frac{n_t(a) - n_t(a+1)}{n_t(a)} * \frac{n_t(a)}{n_t}, \\ &= \sum_{a=0}^{\hat{a}-1} (\hat{a} - (a+1)) * \frac{n_{t-a}(a) - n_{t-a}(a+1)}{n_{t-a}(a)} * S_t(a). \end{aligned}$$

As society S_t is stationary, Lemma 1 applies and we have $S_t(a) = \Pi_{k=0}^{a-1}(1 - \mu_k^t)$ (Eq. (5)). The result follows from the definition of the age-specific mortality rate, i.e. $\mu_a^t = \frac{n_{t-a}(a) - n_{t-a}(a+1)}{n_{t-a}(a)}$.

We then proves that $GD_\theta(S_t) = ED_\theta(S_t)$ for all $\theta \in [0, 1]$. As society S_t is stationary, Lemma 1 applies and we have $N_t = n_t LE_t$ (Eq. (6)). Substituting this expression for N_t into the definition of GD_θ yields the result if we have $YL_t = n_t LGE_{\hat{a}}$.

There remains to show that $YL_t = n_t LGE_{\hat{a}}$. As society S_t is stationary, Lemma 1 applies and we have $\frac{n_{t-a}(a)}{n_t} = \prod_{k=0}^{a-1}(1 - \mu_k^t)$ (Eq. (5)). Substituting this expression for $n_{t-a}(a)$ into the definition of YL_t, which we recall is $YL_t = \sum_{a=0}^{\hat{a}-2} n_{t-a}(a) * \mu_a^t * (\hat{a} - (a+1))$, yields

$$YL_t = n_t \sum_{a=0}^{\hat{a}-2} (\hat{a} - (a+1)) * \mu_a^t * \prod_{k=0}^{a-1} (1 - \mu_k^t),$$

which shows that $YL_t = n_t LGE_{\hat{a}}$ (see Eq. (10) and recall that $\hat{a} - (a+1) = 0$ when $a = \hat{a} - 1$), the desired result.

6 Appendix: Characterization of the ED index

First, we introduce the set-up used in Baland et al. (2021), which we use for the characterization of ED.

Each individual i is associated to a birth year $b_i \in \mathbb{Z}$. In period t, each individual i with $b_i \leq t$ is characterized by a **bundle** $x_i = (a_i, s_i)$, where $a_i = t - b_i$ is the age that individual i would have in period t given her birth year b_i, and s_i is a categorical variable capturing individual status in period t, which can be either alive and non-poor (NP), alive and poor (AP) or dead (D), i.e. $s_i \in S = \{NP, AP, D\}$. In the following, we often refer to individuals whose status is AP as "poor". We consider here that births occur at the beginning while deaths occur at the end of a period. As a result, an individual whose status in period t is D died before period t.[24]

An individual "dies prematurely" if she dies before reaching the minimal lifespan $\hat{a} \in \mathbb{N}$. Formally, period t is "prematurely lost" by any individual i with $s_i = D$ and $a_i < \hat{a}$. A **distribution** $x = (x_1, \ldots, x_{n(x)})$ specifies the age and the status in period t of all $n(x)$ individuals. Excluding trivial distributions for which no individual is

[24] All newborns have age 0 during period t and some among these newborns may die at the end of period t. This implies that $b_i = t \Rightarrow s_i \neq D$.

alive or prematurely dead, the set of distributions in period t is

$$X = \{x \in \cup_{n\in\mathbb{N}}(\mathbb{Z} \times S)^n \mid \text{there is } i \text{ for whom either } s_i \neq D \text{ or } s_i = D \text{ and } \hat{a} > t - b_i\}.$$

Baland et al. (2021) show that the most natural *consistent* index to rank distributions in X is the inherited deprivation index (ID). Let $d(x)$ denote the number of *prematurely dead* individuals in distribution x, which is the number of individuals i for whom $s_i = D$ and $\hat{a} > t - b_i$, $p(x)$ the number of individuals who are poor and $f(x)$ the number of alive and non-poor individuals. The ID index is defined as

$$ID_\theta(x) = \underbrace{\frac{d(x)}{f(x) + p(x) + d(x)}}_{quantity\ deprivation} + \theta \underbrace{\frac{p(x)}{f(x) + p(x) + d(x)}}_{quality\ deprivation}, \tag{13}$$

where $\theta \in [0, 1]$ is a parameter weighing the relative importance of alive deprivation and lifespan deprivation. An individual losing prematurely period t matters $1/\theta$ times as much as an individual spending period t in alive deprivation.

We introduce additional notation for the mortality taking place in period t. Consider the population pyramid in period t, and let $n_a(x)$ be the number of *alive* individuals of age a in distribution x, i.e. the number of individuals i for whom $a_i = a$ and $s_i \neq D$. (The definition of $n_a(x)$ corresponds to $n_{t-a}(a)$ in the notation used in the main text of the paper. In this section, we adopt the notation of Baland et al. (2021), which does not require to mention period t.) The age-specific mortality rate $\mu_a \in [0, 1]$ denotes the fraction of alive individuals of age a dying at the end of period t: the number of a-year-old individuals dying at the end of period t is $n_a(x) * \mu_a$. Letting $a^* \in \mathbb{N}$ stand for the maximal lifespan (which implies $\mu_{a^*-1} = 1$), the **vector of age-specific mortality rates** in period t is given by $\mu = (\mu_0, \ldots, \mu_{a^*-1})$. Vector μ summarizes mortality in period t, while distribution x summarizes alive deprivation in period t as well as mortality before period t. The set of mortality vectors as:

$$M = \left\{\mu \in [0, 1]^{a^*} \middle| \mu_{a^*-1} = 1\right\}.$$

We consider pairs (x, μ) for which the distribution x is a priori unrelated to vector μ. We assume that the age-specific mortality rates μ_a must be feasible given the number of alive individuals $n_a(x)$. Given that distributions have finite numbers of individuals, mortality rates cannot take irrational values, i.e. $\mu_a \in [0, 1] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. The set of pairs considered is

$$O = \left\{(x, \mu) \in X \times M \middle| \text{for all } a \in \{0, \ldots, a^*\} \text{ we have } \mu_a = \frac{c_a}{n_a(x)} \text{ for some } c_a \in \mathbb{N}\right\}.$$

Letting $d_a(x)$ be the number of *dead* individuals born a years before t in distribution x, the total number of individuals born a years before t is then equal to $n_a(x) + d_a(x)$. Formally, the **pair** (x, μ) is **stationary** if, for some $n^* \in \mathbb{N}$ and all $a \in \{0, \ldots, a^*\}$, we have:

- $n_a(x) + d_a(x) = n^* \in \mathbb{N}$ (constant natality),
- $n_{a+1}(x) = n_a(x) * (1 - \mu_a)$ (identical population pyramid in $t + 1$).

In a stationary pair, the population pyramid is such that the size of each cohort can be obtained by applying to the preceding cohort the current mortality rate. The pair associated to a stationary society (as defined in the main text) is stationary.

An index is a function $P : O \times \mathbb{N} \rightarrow \mathbb{R}_+$. We simplify the notation $P(x, \mu, \hat{a})$ to $P(x, \mu)$ as a fixed value for $\hat{a}$ is assumed.

We now introduce the properties characterizing ED. ID Equivalence requires that, as current mortality (in period t) is the same as mortality from previous periods in stationary societies, any index defined on current mortality rates is equivalent to ID in the case of a stationary pair.[25]

Deprivation axiom 1 (ID Equivalence)**.** *There exists some $\theta \in (0, 1]$ such that for all $(x, \mu) \in O$ that are stationary we have $P(x, \mu) = ID_\theta(x)$.*

Independence of Dead requires that past mortality does not affect the index. More precisely, the presence of an additional dead individual in distribution x does not affect the index.

Deprivation axiom 2 (Independence of Dead)**.** *For all $(x, \mu) \in O$ and $i \leq n(x)$, if $s_i = D$, then $P((x_i, x_{-i}), \mu) = P(x_{-i}, \mu)$.*

Independence of Birth Year requires that the index does not depend on the birth year of individuals, i.e. only their status matters. As Independence of Dead requires to disregard dead individuals, the only relevant information in x is whether an alive individual is poor or not.

Deprivation axiom 3 (Independence of Birth Year)**.** *For all $(x, \mu) \in O$ and $i \leq n(x)$, if $s_i = s'_i$, then $P((x_i, x_{-i}), \mu) = P((x'_i, x_{-i}), \mu)$.*

Replication Invariance requires that, if a distribution is obtained by replicating another distribution several times, they both have the same deprivation when associated to the same mortality vector. By definition, a k-replication of distribution x is a distribution $x^k = (x, \ldots, x)$ for which x is repeated k times.

Deprivation axiom 4 (Replication Invariance)**.** *For all $(x, \mu) \in O$ and $k \in \mathbb{N}$, $P(x^k, \mu) = P(x, \mu)$.*

Proposition 6 shows that these properties jointly characterize the ED index.

Proposition 6 (Characterization of ED)**.**
$P = ED_\theta$ if and only if P satisfies Independence of Dead, ID Equivalence, Replication Invariance and Independence of Birth Year.

Proof. We first prove sufficiency. Proving that the ED index satisfies Independence of Dead, Replication Invariance and Independence of Birth Year is straightforward and left to the reader. Finally, ED index satisfies ID Equivalence because ED is equal to GD in stationary populations (Proposition 5) and GD satisfies ID Equivalence (see Baland et al. (2021)). (The pairs associated to stationary societies are stationary).

We now prove necessity. Take any pair $(x, \mu) \in O$. We construct another pair (x''', μ) that is stationary and such that $P(x''', \mu) = P(x, \mu)$ and $ED(x''', \mu) = ED(x, \mu)$. Given that (x''', μ) is stationary, we have by ID Equivalence that $P(x''', \mu) =$

[25] Recall that past mortality is recorded in distribution x while current mortality is recorded in vector μ. As vector μ is redundant in stationary pairs, in the sense that μ can be inferred from the population pyramid, the index can be computed on distribution x only. See Baland et al. (2021) for a complete motivation for this axiom.

$ID_\theta(x''', \mu)$ for some $\theta \in (0, 1]$. As $ID_\theta = GD_\theta = ED_\theta$ for stationary pairs, we have $P(x''', \mu) = ED_\theta(x''', \mu)$ for some $\theta \in (0, 1]$. If we can construct such pair (x''', μ), then $P(x, \mu) = ED_\theta(x, \mu)$ for some $\theta \in (0, 1]$, the desired result.

We turn to the construction of the stationary pair (x''', μ), which will be based on the construction of two intermediary pairs (x', μ) and (x'', μ). One difficulty is to ensure that the mortality rates μ_a ca be achieved in the stationary population given the number of alive individuals $n_a(x''')$, which is $\mu_a = \frac{c}{n_a(x''')}$ for some $c \in \mathbb{N}$.

We first construct a n'−replication of x that has sufficiently many alive individuals to meet this constraint. For any $a \in \{0, \ldots, a^* - 1\}$, take any naturals c_a and e_a such that $\mu_a = \frac{c_a}{e_a}$. Let $e = \prod_{j=0}^{a^*-1} e_j$, $n'_a = e \prod_{j=0}^{a-1}(1 - \frac{c_j}{e_j})$ and $n' = \sum_{j=0}^{a^*-1} n'_j$.[26] Let x' be a n'−replication of x. Letting $n^x = \sum_{j=0}^{a^*-1} n_j(x)$ be the number of alive individuals in distribution x, we have that x' has $n' * n^x$ alive individuals. We have $P(x', \mu) = P(x, \mu)$ by Replication Invariance.

We define x'' from x' by changing the birth years of *alive* individuals in such a way that (x'', μ) has a population pyramid that is stationary. Formally, we construct x'' with $n(x'') = n(x')$ such that

- dead individuals in x' are also dead in x'',
- alive individuals in x' are also alive in x'' and have the same status,
- the birth year of alive individuals are changed such that, for each $a \in \{0, \ldots, a^* - 1\}$, the number of a-years old individuals is $n' * n^x * \frac{\prod_{j=0}^{a-1}(1-\frac{c_j}{e_j})}{\sum_{k=0}^{a^*-1}\prod_{j=0}^{k-1}(1-\frac{c_j}{e_j})}$.[27]

One can check that (x'', μ) has a population pyramid corresponding to a stationary population and that each age group has a number of alive individuals in $\mathbb{N}$. We have $P(x'', \mu) = P(x', \mu)$ by Independence of Birth Year.

Define x''' from x'' by changing the number and birth years of *dead* individuals in such a way that (x''', μ) is stationary. To do so, place exactly $n_0(x'') - n_a(x'')$ dead individuals in each age group a. We have $P(x''', \mu) = P(x'', \mu)$ by Independence of Dead.

Together, we have that $P(x''', \mu) = P(x, \mu)$. Finally, by construction we have $H(x''') = H(x)$, which implies that $ED(x''', \mu) = ED(x, \mu)$. □

[26]These numbers imply that a constant natality of e newborns leads to a stationary population of n' alive individuals.

[27]Observe that $\sum_{k=0}^{a^*-1}\prod_{j=0}^{k-1}(1 - \frac{c_j}{e_j}) = LE$, implying that $e = \frac{n' * n^x}{\sum_{k=0}^{a^*-1}\prod_{j=0}^{k-1}(1-\frac{c_j}{e_j})}$.